Regularity of monoids under Schützenberger products

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Abstract

In this paper we give a partial answer to the problem which is about the regularity of Schützenberger products in semigroups asked by Gallagher in his thesis [3, Problem 6.1.6] and, also, we investigate the regularity for the new version of the Schützenberger product which was defined in [1].

Keywords: The Schützenberger product, regularity.

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1 Introduction and Preliminaries

We recall that a monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that $aba = a$ and $bab = b$ (or, equivalently, for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a \text{ and } bab = b\}$, M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset). In [3, Problem 6.1.6], Gallagher asked whether there exists a classification for arbitrary semigroups A and B for which the Schützenberger product $A\Diamond B$ is regular. In fact, before asking this problem, the question of the regularity of the wreath product of monoids was explained by Skornjakov ([9]). After that, in [6], it has been investigated the regular properties of semidirect and wreath products of monoids. In this paper, to convience the above problem, we purpose to give a partial answer by defining necessary and sufficient conditions of the Schützenberger product $A\Diamond B$ to be regular where both A and B are any monoids. Moreover, by giving a new version of the Schützenberger product ([1]), say $A\Diamond_v B$, we will present another result about this regularity problem.

A generating and defining relation sets for the Schützenberger product of arbitrary monoids have been defined in a joint paper written by Howie and Ruskuc (in [4]). Moreover, in [3], Gallagher defined the finitely generatability and finitely presentability of this product and then he left an open problem explained in the above paragraph.

Let A and B be any monoids with associated presentations $\wp_A = [X; R]$ and $\wp_B = [Y; S]$, respectively. Each paragraph at the rest of this section, we will recall definitions of some products which will be needed for the main results of this paper.

Let $M = A \rtimes_\theta B$ be the corresponding semidirect products of these two monoids, where θ is a monoid homomorphism from B to $End(A)$ such that, for every $a \in A$, $b_1, b_2 \in B$, $(a)\theta_{b_1 b_2} = ((a)\theta_{b_2})\theta_{b_1}$. We recall that the elements of M can be regarded as ordered pairs (a, b), where $a \in A$, $b \in B$ with the multiplication given by $(a_1, b_1)(a_2, b_2) = (a_1(a_2)\theta_{b_1}, b_1 b_2)$, and the monoids A and B are identified with the submonoids of M having elements $(a, 1_B)$ and $(1_A, b)$. For every $x \in X$ and $y \in Y$, choose a word, denoted by $(x)\theta_y$, on X such that $[(x)\theta_y] = [x]\theta_{[y]}$ as an element of K. To establish notation, let us denote the relation $yx = (x)\theta_y y$ on $X \cup Y$ by T_{yx} and write T for the set of relations T_{yx}. Then, for any choice of the words $(x)\theta_y$, $\wp_M = [X, Y\ ;\ R, S, T]$ is a standard monoid presentation for the semidirect product M.

The cartesian product of B copies of the monoid A is denoted by $A^{\times B}$, while the corresponding direct product is denoted by $A^{\oplus B}$. One may think of $A^{\times B}$ as the set of all such functions from B to A, and $A^{\oplus B}$ as the set all such functions f having finite support, that is to say, having the property that $(x)f = 1_A$ for all but finitely many x in B. The unrestricted and restricted wreath products of the monoid A by the monoid B, are the sets $A^{\times B} \times B$ and $A^{\oplus B} \times B$, respectively, with the multiplication defined by $(f, b)(g, b') = (f\ {}^b g, bb')$, where ${}^b g : B \to A$ is defined by

$$(x)^b g = (xb)g, \quad (x \in B) \tag{1}$$

such that $(xb)g$ has finite support. It is well known that both these wreath products are monoids with the identity $(\overline{1}, 1_B)$, where $x\overline{1} = 1_A$ for all $x \in B$. (For more details on the definition and applications of restricted (unrestricted) wreath products, we can refer, for instance, [2, 4, 5, 8, 7]). We should note that, for having finite support, B must be finite or groups.

Now for a subset P of $A \times B$ and $a \in A$, $b \in B$, we let define

$$Pb = \{(c, db)\ ;\ (c, d) \in P\} \ \text{ and } \ aP = \{(ac, d)\ ;\ (c, d) \in P\}.$$

Then the Schützenberger product of A and B, denoted by $A\Diamond B$, is the set $A\times P(A\times B)\times B$ with the multiplication $(a_1,P_1,b_1)(a_2,P_2,b_2) = (a_1a_2, P_1b_2\cup a_1P_2, b_1b_2)$. Clearly $A\Diamond B$ is a monoid ([4]) with the identity $(1_A,\emptyset,1_B)$.

2 Main Theorems

The following first theorem aims to give necessary and sufficient conditions for $A\Diamond B$ to be regular while both A and B are arbitrary monoids.

Theorem 2.1 *Let A and B be any monoids. The product $A\Diamond B$ is regular if and only if*

(i) A and B are regular,

(ii) for every $(a,P,b)\in A\Diamond B$, either

$$P = aP_1b = \bigcup_{(a_1,b_1)\in P_1}\{(aa_1, b_1b)\} \quad or \quad P = caP_1bd = \bigcup_{(a_1,b_1)\in P_1}\{(caa_1, b_1bd)\},$$

where $P_1\subseteq A\times B$ and $c\in a^{-1}$, $d\in b^{-1}$.

By (1) and the definiton of Schützenberger product, we can define a new version of the Schützenberger product as follows. We note that the definition and some other properties of this product have been investigated in [1].

Let A and B be monoids. We recall that $A^{\oplus B}$ is the set of all functions f having finite support. For $P\subseteq A^{\oplus B}\times B$ and $b\in B$, we define the set

$$Pb = \{(f,db); (f,d)\in P\}.$$

The new version of the Schützenberger product of A by B, denoted by $A\Diamond_v B$, is the set $A^{\oplus B}\times P(A^{\oplus B}\times B)\times B$ with the multiplication

$$(f,P_1,b_1)(g,P_2,b_2) = (f\ {}^{b_1}g, P_1b_2\cup P_2, b_1b_2).$$

One can easily show that $A\Diamond_v B$ is a monoid with the identity $(\overline{1},\emptyset,1_B)$, where ${}^{b_1}g$ is defined as in (1). We should also note that, for having finite support, B must be finite or groups.

Thus another main result of this paper is the following.

Theorem 2.2 *Let A be an arbitrary monoids and B be a finite monoid or be a group. Then $A\Diamond_v B$ is regular if and only if*

(*i*) *A and B are regular,*

(*ii*) *For every* $x \in B$ *and* $f \in A^{\oplus B}$ *there exist* $e \in B$ *such that* $e^2 = e$, *with*

$$(x)f \in A(xe)f.$$

(*iii*) *for every* $(f, P, b) \in A\Diamond_v B$, *either*

$$P = P_1 b = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 b)\} \quad or \quad P = P_1 bd = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 bd)\},$$

where $P_1 \subseteq A^{\oplus B} \times B$ *and* $d \in b^{-1}$.

3 Proofs

Proof of Theorem 2.1: Let us suppose that $A\Diamond B$ is regular. Thus, for $(a, \emptyset, b) \in A\Diamond B$, there exists (c, P, d) such that

$$\begin{aligned} (a, \emptyset, b) &= (a, \emptyset, b)(c, P, d)(a, \emptyset, b) = (aca, aPb, bdb), \\ (c, P, d) &= (c, P, d)(a, \emptyset, b)(c, P, d) = (cac, Pbd \cup caP, dbd). \end{aligned}$$

Therefore we have $a = aca$, $c = cac$, $b = bdb$ and $d = dbd$. This implies that (i) must hold.

By the assumption on the regularity of $A\Diamond B$, for $(a, P, b) \in A\Diamond B$, we have $(c, P_2, d) \in A\Diamond B$ such that

$$(a, P, b) = (a, P, b)(c, P_2, d)(a, P, b) \quad \text{and} \quad (c, P_2, d) = (c, P_2, d)(a, P, b)(c, P_2, d).$$

Hence this gives us $a = aca$, $c = cac$, $b = bdb$, $d = dbd$, $P = Pdb \cup aP_2b \cup acP$ and $P_2 = P_2bd \cup cPd \cup caP_2$. To show the second condition in theorem, let us suppose that $P \neq aP_1b$, for some $P_1 \subseteq A \times B$. Then there exists $(a_2, b_2) \in P$ such that $a_2 \neq aa_2'$ and $b_2 \neq b_2'b$ where $a_2' \in A$ and $b_2' \in B$. Thus P can not be equal to $Pdb \cup aP_2b \cup acP$, for all $P_2 \subseteq A \times B$. This gives a contradiction with the regularity of $A\Diamond B$. In fact, when someone take $P = aP_1b$, the equalities

$$\begin{aligned} Pdb \cup aP_2b \cup acP = aP_1bdb \cup aP_2b \cup acaP_1b &= aP_1b \cup aP_2b \cup aP_1b \\ &= aP_1b \quad \text{by choosing } P_2 = caP_1bd \\ &= P \end{aligned}$$

and

$$
\begin{aligned}
P_2bd \cup cPd \cup caP_2 &= P_2bd \cup caP_1bd \cup caP_2 \\
&= caP_1bdbd \cup caP_1bd \cup cacaP_1bd \\
&\quad \text{by choosing } P_2 = caP_1bd \\
&= caP_1bd \cup caP_1bd \cup caP_1bd = caP_1bd = P_2
\end{aligned}
$$

hold. We note that, by applying similar discussions as above for the case $P = caP_1bd$ in theorem, where $P_1 \subseteq A \times B$ and $c \in a^{-1}$, it is seen that condition (ii) must hold.

For the converse part of the proof, let $(a, P, b) \in A\Diamond B$. Thus we definitely have $c \in A$ and $d \in B$ such that $c \in a^{-1}$ and $d \in b^{-1}$. Now let us consider the union of sets

$$Pdb \cup aP_2b \cup acP \quad \text{and} \quad P_2bd \cup cPd \cup caP_2.$$

At this stage, by $P = aP_1b$, if we choose $P_2 = caP_1bd \subseteq A \times B$, then we get

$$Pdb \cup aP_2b \cup acP = aP_1b = P \quad \text{and} \quad P_2bd \cup cPd \cup caP_2 = caP_1bd = P_2.$$

As a result of this, for every $(a, P, b) \in A\Diamond B$, there exists $(c, P_2, d) \in A\Diamond B$ such that

$$
\begin{aligned}
(a, P, b)(c, P_2, d)(a, P, b) &= (aca, Pdb \cup aP_2b \cup acP, bdb) = (a, P, b), \\
(c, P_2, d)(a, P, b)(c, P_2, d) &= (cac, P_2bd \cup cPd \cup caP_2, dbd) = (c, P_2, d).
\end{aligned}
$$

In addition, by applying similar above arguments for the case $P = caP_1bd$ in theorem, where $P_1 \subseteq A \times B$ and $c \in a^{-1}$, the proof of the regularity of $A\Diamond B$ is completed.

Hence the result. $\square$

Proof of Theorem 2.2: Let us suppose that $A\Diamond_v B$ is regular. Thus, for $(f, (1_A, 1_B), b) \in A\Diamond_v B$, there exists $(g, P, d) \in A\Diamond_v B$ such that

$$
\begin{aligned}
(f, (1_A, 1_B), b) &= (f, (1_A, 1_B), b)(g, P, d)(f, (1_A, 1_B), b), \\
(g, P, d) &= (g, P, d)(f, (1_A, 1_B), b)(g, P, d).
\end{aligned}
$$

We then have $b = bdb$ and $d = dbd$. If we choose $b = 1$ then we have $bd = 1$. Therefore we have $f = fgf$ and $g = gfg$. This implies that both B and $A^{\oplus B}$ are regular. Since $A^{\oplus B}$ denotes the direct product of B copies of A, it is easy to see that if $A^{\oplus B}$ is regular, then A is regular. This gives condition (i).

By the assumption, for every $(f, P, b) \in A\Diamond_v B$, we have $(g, P_2, d) \in A\Diamond_v B$ such that

$$\begin{aligned}(f, P, b) &= (f, P, b)(g, P_2, d)(f, P, b) = (f\ {}^{b}g\ {}^{bd}f, Pdb \cup P_2 b \cup P, bdb),\\ (g, P_2, d) &= (g, P_2, d)(f, P, b)(g, P_2, d) = (g\ {}^{d}f\ {}^{db}g, P_2bd \cup Pd \cup P_2, dbd).\end{aligned}$$

Hence, by equating the components, we get $f = f\ {}^{b}g\ {}^{bd}f$, $g = g\ {}^{d}f\ {}^{db}g$, $b = bdb$, $d = dbd$, $P = Pdb \cup P_2b \cup P$ and $P_2 = P_2bd \cup Pd \cup P_2$. These show that, for every $x \in B$,

$$(x)f \ = \ (x)f\ (x)^{b}g\ (x)^{bd}f = (x)f\ (xb)g\ (xbd)f \in A(xbd)f.$$

If we take $e = bd$, then condition (ii) becomes true. In addition, by using the facts $b = bdb$, $d = dbd$, $P = Pdb \cup P_2b \cup P$ and $P_2 = P_2bd \cup Pd \cup P_2$, for every $(f, P, b) \in A\Diamond_v B$, and by applying similar arguments given in the proof of Theorem 2.1, we get

$$\text{either} \quad P = P_1b \quad \text{or} \quad P = P_1bd,$$

where $P_1 \subseteq A^{\oplus B} \times B$ and $d \in b^{-1}$. Therefore condition (iii) must hold.

Conversely, let us suppose that the monoids A and B satisfy conditions (i), (ii) and (iii). For $x, b, d \in B$ and $f, g \in A^{\oplus B}$, we let consider

$$(x)f\ (x)^{b}g\ (x)^{bd}f,$$

where $dbd = d$. By condition (ii), for $a \in A$, we have $(x)f = a(xbd)f$ where $bd = e$. Thus

$$(x)f\ (x)^{b}g\ (x)^{bd}f = a(xbd)f\ (x)^{b}g\ (x)^{bd}f = a(x)^{bd}f\ (x)^{b}g\ (x)^{bd}f. \tag{2}$$

Since A is regular, $A^{\oplus B}$ is regular [6]. Thus we can choose $g = {}^{d}v$ $(v \in A^{\oplus B})$ such that $fvf = f$ and $vfv = v$. Hence the last term in (2) will be equal to

$$a(x)^{bd}f\ (x)^{bd}v\ (x)^{bd}f = a(x)^{bd}(fvf) = a(x)^{bd}f = (x)f.$$

This implies that $f = f\ {}^{b}g\ {}^{bd}f$. On the other hand, by similar procedure as above, we obtain

$$g\ {}^{d}f\ {}^{db}g = {}^{d}v\ {}^{d}f\ {}^{dbd}v = {}^{d}v\ {}^{d}f\ {}^{d}v = {}^{d}(vfv) = {}^{d}v = g.$$

Moreover, by condition (iii), we have $P = P_1b$ or $P = P_1bd$, where $P_1 \subseteq A^{\oplus B} \times B$. For the next stage of proof, we will only consider $P = P_1b$ since similar progress can be applied for the other value of P. Therefore there exists a subset $P_2 = P_1bd$ of $A^{\oplus B} \times B$ such that

$$\begin{aligned}Pdb \cup P_2b \cup P &= P_1bdb \cup P_1bdb \cup P_1b = P_1b \cup P_1b \cup P_1b = P_1b = P,\\ P_2bd \cup Pd \cup P_2 &= P_1bdbd \cup P_1bd \cup P_1bd = P_1bd \cup P_1bd \cup P_1bd = P_1bd = P_2.\end{aligned}$$

As a result of these above procedure, for every $(f,P,b)\in A\Diamond_v B$, there exists $(g,P_2,d)\in A\Diamond_v B$ such that

$$(f,P,b)(g,P_2,d)(f,P,b) = (f\,{}^b g\,{}^{bd} f, Pdb\cup P_2 b\cup P, bdb) = (f,P,b),$$
$$(g,P_2,d)(f,P,b)(g,P_2,d) = (g\,{}^d f\,{}^{db} g, P_2 bd\cup Pd\cup P_2, dbd) = (g,P_2,d).$$

Hence the result. □

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